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Wave propagation in a bifurcated impedance-lined cylindrical waveguide

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Abstract The radiation of a cylindrical-surface-wave mode which propagates towards the mouth of a semi-infinite cylindrical waveguide which supports surface waves is considered. This semi-infinite cylindrical waveguide is symmetrically located inside an infinite cylindrical waveguide whose surfaces are lined with an absorbent material. The whole system constitutes a new bifurcated cylindrical-waveguide boundary-value problem that has application in acoustics and electromagnetism. The mathematical model results in a scalar Wiener–Hopf problem which can be rigorously solved to give a closed-form solution.

Keywords Cylindrically lined ducts · Diffraction · Lined waveguides · Wiener–Hopf technique

1 Introduction

The bifurcated waveguide problem under consideration here is shown in Fig. 1. A surface-wave mode is assumed to propagate towards the mouth of a lined semi-infinite cylindrical waveguide. The boundary-value problem which we are going to solve in this paper is of a fairly general nature. The boundary conditions on all the cylindrical surfaces are of the third kind, that is, Robin type. There is a number of practical physical situations where this waveguide system with these boundary conditions can arise. For example, by using various polarizations of the surface wave, the present scalar problem arises as a model for electromagnetic communication in subterranean tunnels. Indeed, the lossy-impedance condition on the inner wall of the larger cylinder models real tunnel conditions quite well [1] and, in this context, the semi-infinite waveguide would be viewed as a wave-launcher [2]. The present problem can also be used to model the propagation of waves in fibre-optic waveguides with the surface-impedance boundary conditions modelling a metal–dielectric cladding of a fibre-optic waveguide [3,4]. In acoustics, the design of exhaust and ventilation systems that reduce unwanted noise use absorbent linings along cylindrical ducts. The attenuation of unwanted sound in infinite closed ducts by means of acoustically absorbing liners has been theoretically analyzed extensively in the literature; see [5,6] and the review articles [7,8]. More recently, Büyükaksoy and Demir have published a series of articles that bear on this subject area; in particular, in [9] they solved a related problem “by the modified Wiener–Hopf techique”. They considered the same cylindrical geometry but with different absorbent cylindrical surfaces. The solution of this quite general problem by the modified Wiener–Hopf technique resulted in

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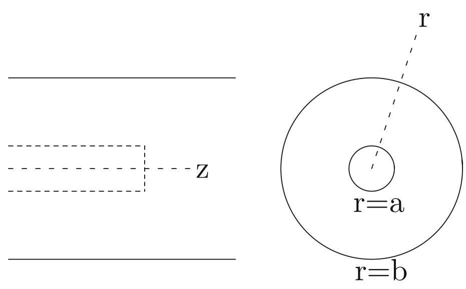

Fig. 1 Geometry of the cylindrical duct system

an infinite system of algebraic equations that was solved numerically to give an approximate solution. The present work could be used as a bench-mark for a test on their approximate numerical solutions.

Here, to be specific, we shall couch the present paper in an acoustic context. Thus, an acoustic surface wave propagates towards the mouth of the semi-infinite duct. The boundary conditions on the internal surfaces of the semi-infinite waveguide are such that this guide supports surface waves. The infinite cylindrical duct in which the semi-infinite guide is situated is such that the walls are acoustically absorbent. Thus, this particular problem can be considered as a system where the surface wave emitted from the semi-infinite region is unwanted noise and the absorbent lining is used to reduce the noise down the duct far away from the source of noise. Alternatively the semi-infinite guide can be considered as a measuring probe that is placed inside a larger lined duct to measure the reflection coefficient of the surface wave reflected back into the semi-infinite duct. The reflection coefficient will depend on the impedance of the infinite duct lining. Such a device has practical applications in large industrial chimneys where soot lines the chimney surface and therefore changes the surface impedance [10]. The reflection coefficient will give an indication of how clogged up the chimney is and whether it needs to be cleaned. A related situation arises in blocked-up arteries of the human body, and the present problem could be used to build an instrument to indicate the degree of clogging of the blood vessels. Clearly, the artery walls being modelled by impedance boundary conditions and being cylindrically rigid is a simplification of the real-life situation. Even so, it offers a first prototype model for a more refined model.

In Sect. 2 we shall formulate the mathematical problem that we intend to solve. In Sect. 3 we shall solve the problem formulated in Sect. 2 by means of the Wiener–Hopf technique. The solution will be expressed as complex contour integrals. In Sect. 4 we shall analytically convert these integrals into infinite series of modes which propagate in the waveguide region. Graphs will be given for the reflection coefficient which is a useful way of measuring power and also the effect of the lining impedance of the infinite duct wall. At the end of this work we shall present some appendices that derive analytical details and calculations that are required in the main body of the paper.

2 Formulation of the boundary-value problem

We shall consider the acoustic diffraction of a wave mode propagating towards the open end of a semi-infinite cylindrical tube whose internal surface is capable of supporting a surface-wave mode. This semi-infinite tube is surrounded by an infinite cylindrical casing which is lined with an acoustically absorbing, or wave bearing material. The cylinder casing and its lining are located at $r = b$, $-\infty < z < \infty$, and the semi-infinite cylinder, which is assumed to be infinitely thin, is located at $r = a$, $-\infty < z < 0$, in cylindrical polar coordinates (r, θ, z) as shown in Fig. 1. The sound source emanates cylindrically symmetric modes, from $z = -\infty$, along the inside(or outside) of the semi-infinite tube towards the open end at $z = 0$. Therefore, the source field may be represented as a sum of symmetric wave modes that are independent of θ. Such a situation arises, for example, when the source field is a point (or ring) source located at $(0, 0, z_0)$ (or (c, θ_0, z_0)), $a < c < b$, $0 \leq \theta_0 < 2\pi$), $z_0 \ll 0$. From the geometrical symmetry of the problem in relation to the incident field, the total acoustic field everywhere will be independent of θ. We shall therefore introduce a scalar potential function $\psi(r, z, t)$ which defines the acoustic pressure and velocity for an ideal compressible irrotational fluid by $p = -\rho_0 \frac{\partial \psi}{\partial t}$, and $\mathbf{u} = \nabla \psi$, respectively, where ρ_0 is the density of the undisturbed medium. The incident sound field is assumed to have a time-harmonic variation and therefore the field everywhere can be represented by $\psi(r, z; t) = \mathfrak{Re}[e^{-\mathrm{i}\omega t}\phi(r, z)]$. We shall not show the time variation $\mathrm{e}^{-\mathrm{i}\omega t}$ in the rest of the paper and only work with the complex potential function $\phi(r, z)$. For a surface which absorbs acoustic energy it is necessary to describe mathematically how the acoustic energy is transmitted into the boundary surface. In many practical situations the so-called locally reacting surface proves to be a good model for porous absorbing surfaces; see [11] and [12, pp. 151–179]. The normal acoustic impedance Z_n of the surface of such an absorbent lining is defined by the ratio $Z_n = p/(\mathbf{u} \cdot \mathbf{n})$ where the unit normal $\mathbf{n}$ is directed into the absorbent lining. Thus, in terms of the complex velocity potential ϕ, the boundary condition on the absorbent surface is given by

$$\frac{\partial \phi}{\partial n} - \frac{\mathrm{i}k}{\zeta}\phi = 0, \tag{1}$$

where $\zeta(= Z_n/(\rho_0 c))$ is the complex specific impedance, with positive real part, c is the speed of sound, and $k = \omega/c$ is the wave number. Wave-bearing corrugated surfaces which do not absorb energy can be described by a similar type of boundary condition; see [13]. In this situation the complex specific surface impedance $\zeta = \chi - \mathrm{i}\xi$ has a zero resistive component $\chi = 0$, and a purely negative reactance $\xi < 0$. The quantity ξ depends on the geometry of the corrugated surface. Thus, the boundary-value problem satisfied by $\phi(x, z)$ is given by:

$$(\nabla^2 + k^2)\phi(x, z) = 0, \quad (0 < r < a) \cup (a < r < b); \tag{2}$$

$$\frac{\partial \phi}{\partial r}(a^+, z) + \frac{k}{\xi}\phi(a^+, z) = \frac{\partial \phi}{\partial r}(a^-, z) + \frac{k}{\xi}\phi(a^-, z) = 0, \quad (z < 0); \tag{3}$$

$$\frac{\partial \phi}{\partial r}(a^+, z) + \frac{k}{\xi}\phi(a^+, z) = \frac{\partial \phi}{\partial r}(a^-, z) + \frac{k}{\xi}\phi(a^-, z), (-\infty < z < \infty); \tag{4}$$

$$\left(\frac{\partial}{\partial r} - \frac{\mathrm{i}k}{\zeta}\right)\phi(b^-, z) = 0, \quad (-\infty < z < \infty); \tag{5}$$

$$\phi(a^-, z) = \phi(a^+, z), \quad (z > 0), \tag{6}$$

where it is assumed $b > a$, and $\mathfrak{Re}\,\zeta \geq 0$ and $\xi < 0$.[1] To the above conditions we add the incident field and those conditions at infinity which are relevant to the nature of the propagating modes that the various duct regions can sustain. It is not difficult to show, by using the method of separation of variables [14] that the appropriate modal expansions in the various duct regions at infinity are stated below.

For $z \to -\infty, 0 \leq r \leq a$:

$$\phi(r, z) = I_0(\mu_0 r)\mathrm{e}^{\mathrm{i}\chi_0 z} + R I_0(\mu_0 r)\mathrm{e}^{-\mathrm{i}\chi_0 z} + \sum_{n=1}^{\infty} R_n J_0(\alpha_n r)\mathrm{e}^{-\mathrm{i}\chi_n z}, \tag{7}$$

where the first and second term correspond to the dominant incident and reflected surface wave, respectively; here μ_0 is *the* positive real solution of the equation

$$\mu_0 I_1(\mu_0 a) + \frac{k}{\xi} I_0(\mu_0 a) = 0, \quad \xi < 0, \tag{8}$$

and $\chi_0 = \sqrt{k^2 + \mu_0^2}$ with $0 < k < k_0$, where k_0 is the smallest positive root of the equation $\xi J_0'(k_0 a) + J_0(k_0 a) = 0$. The remaining terms correspond to the reflected modes where $\alpha_n = \sqrt{k^2 - \chi_n^2}$, $(n = 1, 2, 3, \ldots)$, are the real positive roots of the equation

$$\alpha_n J_0'(\alpha_n a) + \frac{k}{\xi} J_0(\alpha_n a) = 0, \tag{9}$$

[1] The condition $\mathfrak{Re}\,\zeta \geq 0$ corresponds to a situation where the internal surface of the infinite waveguide has an absorbent lining. The condition $\xi < 0$ corresponds to the situation where the internal surface of the semi-infinite waveguide has a lining which supports surface waves.

with $0 < \mathfrak{Im}\chi_1 < \mathfrak{Im}\chi_2 < \mathfrak{Im}\chi_3 < \ldots$, and $\mathfrak{Re}\ \chi_n \geq 0$, $\chi_n = \mathrm{i}\sqrt{\alpha_n^2 - k^2}$, and $k < \alpha_1 < \alpha_2 < \alpha_3 < \ldots$. The proofs that only one surface-wave mode χ_0 can propagate for $0 < k < k_0$, and that the Eq. (9) has an infinite number of real roots is given in the Appendix A.

For $z \to \infty$, $0 \leq r \leq b$:

$$\phi(r, z) = \sum_{n=1}^{\infty} T_n J_0(\beta_n r)\mathrm{e}^{\mathrm{i}\xi_n z}, \tag{10}$$

where $\beta_n = \sqrt{k^2 - \xi_n^2}(n = 1, 2, 3, \ldots)$ are the roots of the equation

$$\beta_n J_1(\beta_n b) + \frac{\mathrm{i}k}{\zeta} J_0(\beta_n b) = 0. \tag{11}$$

It is shown in the Appendix B that $0 < \mathfrak{Im}\xi_1 < \mathfrak{Im}\xi_2 < \mathfrak{Im}\xi_3 < \cdots$, $0 < \mathfrak{Re}\xi_n$, and $\mathfrak{Re}\ \beta_n \mathfrak{Im}\beta_n < 0$.

For $z \to -\infty$, $a \leq r \leq b$

$$\phi(r, z) = \textstyle\sum_{n=1}^{\infty} \overline{T}_n \left[(\delta_n H_0^{(1)\prime}(\delta_n a) + \frac{k}{\xi} H_0^{(1)}(\delta_n a)) J_0(\delta_n r) - (\delta_n J_0'(\delta_n a) + \frac{k}{\xi} J_0(\delta_n a)) H_0^{(1)}(\delta_n r) \right] e^{-i\eta_n z}, \tag{12}$$

where $\delta_n = \sqrt{k^2 - \eta_n}(n = 1, 2, 3, \ldots)$, are the roots of the equation

$$\begin{aligned}(\delta_n H_0^{(1)\prime}(\delta_n a) &+ \frac{k}{\xi} H_0^{(1)}(\delta_n a))(\delta_n J_0'(\delta_n b) - \frac{\mathrm{i}k}{\zeta} J_0(\delta_n b)) - (\delta_n J_0'(\delta_n a) \\ &+ \frac{k}{\xi} J_0(\delta_n a))(\delta_n H_0^{(1)\prime}(\delta_n b) - \frac{\mathrm{i}k}{\zeta} H_0^{(1)}(\delta_n b)) = 0,\end{aligned} \tag{13}$$

with $0 < \mathfrak{Im}\eta_1 < \mathfrak{Im}\eta_2 < \mathfrak{Im}\eta_3 \ldots$, and $0 < \mathfrak{Re}\eta_n$, $\mathfrak{Re}\eta_n \mathfrak{Im}\eta_n > 0$; see the Appendices A and B.

Finally, we require ϕ to have finite energy density on $z = 0$, $r = a$, and thus ϕ must be finite and $|\nabla\phi|$ must have an integrable singularity. This results in the following edge-field behavior at $r = a$, $z \to 0$:

$$\phi(a, z) = O(1), \quad \left| \frac{\partial\phi(a, z)}{\partial r} \right| = O(z^{-\frac{1}{2}}), \quad z \to 0. \tag{14}$$

The satisfaction of the above conditions (2) to (14) will result in a unique solution to the boundary-value problem formulated.

3 Solution of the boundary-value problem

A suitable representation for the total field $\phi(r, z)$ in all space $-\infty < z < \infty$, $r < b$ which satisfies (2) is given by

$$\phi(r, z) = I_0(\mu_0 r)\mathrm{e}^{\mathrm{i}\chi_0 z} + \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu z} A(\nu) J_0(\kappa r)\mathrm{d}\nu, \quad (r < a); \tag{15}$$

$$\phi(r, z) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu z}[B(\nu) J_0(\kappa r) + C(\nu) H_0^{(1)}(\kappa r)]\mathrm{d}\nu, \quad (a < r < b), \tag{16}$$

where $\kappa = \sqrt{k^2 - \nu^2}$. The branch cuts are from k to $\mathrm{i}\infty$ and from $-k$ to $-\mathrm{i}\infty$. The cut Riemann sheet on which we shall work is defined by $0 \leq \arg\kappa \leq \pi$. The contour of integration is indented below the point $-k$ and above the point k.[2] The quantities $A(\nu)$, $B(\nu)$,and $C(\nu)$ are as yet unknown; however, the edge condition (14) requires that as $|\nu| \to \infty$

$$A(\nu) = O(|\nu|^{-1}\mathrm{e}^{-a|\nu|}), \quad \frac{\mathrm{e}^{a|\nu|}}{\sqrt{\pi}} B(\nu) + \sqrt{\pi}\mathrm{e}^{-a|\nu|} C(\nu) = O(|\nu|^{-1}); \tag{17}$$

[2] A posteriori, branch-cut singularities should not arise for physical reasons. We are dealing with a closed-duct system and only pole singularities can occur in the integrands, giving rise to modes of propagation.

see Appendix C. We shall see later that the integrands of (15) and (16) only have pole singularities that produce exponential wave modes propagating down the ducts. These wave modes, in the various duct regions, must be of the form given in the mode conditions at infinity as defined by the expressions (7), (10), and (12). It is shown in the Appendices A and B that none of these poles lie on a suitably indented contour of integration in (15) and (16). To simplify the future resulting equations, we shall use the following formulaic abbreviations:

$$J(\alpha, \beta, \nu) = \kappa J_0'(\kappa\alpha) + k\beta J_0(\kappa\alpha), \tag{18}$$

$$H(\alpha, \beta, \nu) = \kappa H_0^{(1)\prime}(\kappa\alpha) + k\beta H_0^{(1)}(\kappa\alpha). \tag{19}$$

Thus, to determine $A(\nu)$, $B(\nu)$ and $C(\nu)$, we substitute (15) and (16) in the remaining boundary conditions (2) to (6), giving the following expressions:

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu z}[B(\nu)J(a, 1/\xi, \nu) + C(\nu)H(a, 1/\xi, \nu)]\mathrm{d}\nu = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu z} A(\nu)J(a, 1/\xi, \nu)\mathrm{d}\nu = 0, \tag{20}$$

for $(z < 0)$;

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu z}[B(\nu)J(a, 1/\xi, \nu) + C(\nu)H(a, 1/\xi, \nu)]\mathrm{d}\nu = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu z} A(\nu)J(a, 1/\xi, \nu)\mathrm{d}\nu, \tag{21}$$

for $(-\infty < z < \infty)$;

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu z}[B(\nu)J(b, -\mathrm{i}/\zeta, \nu) + C(\nu)H(b, -\mathrm{i}/\zeta, \nu)]\mathrm{d}\nu = 0, \tag{22}$$

for $(-\infty < z < \infty)$;

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu z}\left[\frac{I_0(\mu_0 a)}{2\pi i(\nu - \chi_{0\smile})} + (A(\nu) - B(\nu))J_0(\kappa a) - C(\nu)H_0^{(1)}(\kappa a)\right]\mathrm{d}\nu = 0, \tag{23}$$

for $(z > 0)$. The smile on χ_0 denotes that the contour of integration is indented below the point $\nu = \chi_0$. A solution of the above system of equations can be written as

$$A(\nu)J(a, 1/\xi, \nu) = B(\nu)J(a, 1/\xi, \nu) + C(\nu)H(a, 1/\xi, \nu) = \Phi^-(\nu), \tag{24}$$

$$B(\nu)J(b, -\mathrm{i}/\zeta, \nu) + C(\nu)H(b, -\mathrm{i}/\zeta, \nu) = 0, \tag{25}$$

$$\frac{I_0(\mu_0 a)}{2\pi\mathrm{i}(\nu - \chi_0)} + (A(\nu) - B(\nu))J_0(\kappa a) - C(\nu)H_0^{(1)}(\kappa a) = \Phi^+(\nu), \tag{26}$$

where $\Phi^+(\nu)(\Phi^-(\nu))$ is holomorphic in $\Im\mathfrak{m}\nu \geq 0(\Im\mathfrak{m}\nu \leq 0, \nu \neq \chi_0)$, respectively. By eliminating $A(\nu)$, $B(\nu)$, and $C(\nu)$ from the Eqs. (24)–(26), we get the scalar Wiener–Hopf equation:

$$\frac{I_0(\mu_0 a)}{2\pi\mathrm{i}(\nu - \chi_0)} + K(\nu)\Phi^-(\nu) = \Phi^+(\nu), \tag{27}$$

where

$$K(\nu) = \left(\frac{2}{\mathrm{i}\pi a}\right)\frac{J(b, -\mathrm{i}/\zeta, \nu)}{J(a, 1/\xi, \nu)D(\nu)}, \tag{28}$$

with

$$D(\nu) = J(a, 1/\xi, \nu)H(b, -\mathrm{i}/\zeta, \nu) - J(b, -\mathrm{i}/\zeta, \nu)H(a, 1/\xi, \nu). \tag{29}$$

In order to be able to solve the Wiener–Hopf equation (27) we shall require the following asymptotic growth estimates as $\nu \to \pm\infty$.

$$\kappa = \mathrm{i}|\nu|, \quad J_0(\kappa a) = O(|\nu|^{-\frac{1}{2}}\mathrm{e}^{a|\nu|}), \quad J_0'(\kappa a) = O(|\nu|^{-\frac{1}{2}}\mathrm{e}^{a|\nu|}),$$

$$H_0^{(1)}(\kappa a) = O(|\nu|^{-\frac{1}{2}}\mathrm{e}^{-a|\nu|}), \quad H_0^{(1)\prime}(\kappa a) = O(|\nu|^{-\frac{1}{2}}\mathrm{e}^{-a|\nu|}),$$

$$J(b, -\mathrm{i}/\zeta, \nu) = O(|\nu|^{\frac{1}{2}}\mathrm{e}^{b|\nu|}), \quad J(a, 1/\xi, \nu) = O(|\nu|^{\frac{1}{2}}\mathrm{e}^{a|\nu|}),$$

$$H(b, -\mathrm{i}/\zeta, \nu) = O(|\nu|^{\frac{1}{2}}\mathrm{e}^{-b|\nu|}), \quad H(a, 1/\xi, \nu) = O(|\nu|^{\frac{1}{2}}\mathrm{e}^{-a|\nu|}),$$

$$K(\nu) = O(|\nu|^{-1}), \quad D(\nu) = O(|\nu|\mathrm{e}^{(b-a)|\nu|}).$$

These asymptotic estimates together with (17) and (24)–(26) give, as $\nu \to \pm\infty$,

$$\Phi^-(\nu) = O(|\nu|^{-\frac{1}{2}}), \quad \text{for} \quad \mathfrak{Im}\nu < 0, \tag{30}$$

$$\Phi^+(\nu) = O(|\nu|^{-1}), \quad \text{for} \quad \mathfrak{Im}\nu > 0. \tag{31}$$

By carrying out the factorization

$$K(\nu) = K_+(\nu)K_-(\nu), \tag{32}$$

where the subscripts $\pm$ denote the regions where the factors are holomorphic, we can rewrite Eq. (27) in the form

$$\frac{I_0(\mu_0 a)}{2\pi\mathrm{i}(\nu-\chi_0)}K_+^{-1}(\chi_0) + K_-(\nu)\Phi^-(\nu) = \Phi^+(\nu)K_+^{-1}(\nu) - \frac{I_0(\mu_0 a)}{2\pi\mathrm{i}(\nu-\chi_0)}(K_-^{-1}(\nu) - K_-^{-1}(\chi_0)), \tag{33}$$

which is valid along the real indented line $\mathfrak{Im}\nu = 0$. The detailed factorization of $K(\nu)$, defined by (28) and (32), is carried out in Appendix D. In particular, it is shown there that, as $|\nu| \to \infty$,

$$K_\pm(\nu) = O(|\nu|^{-\frac{1}{2}}), \quad \text{for} \quad \mathfrak{Im}\nu \gtrless 0. \tag{34}$$

By using the asymptotic expressions (31), (32) and (34), we can show that the left-hand side of the (33) is holomorphic and asymptotic to $O(|\nu|^{-1})$ as $|\nu| \to \infty$, in $\mathfrak{Im}\nu \le 0$. Similarly, the right-hand side is holomorphic and asymptotic to $O(|\nu|^{-\frac{1}{2}})$ as $|\nu| \to \infty$, in $\mathfrak{Im}\nu \ge 0$. Hence, by an application of Liouville's theorem, the function which is the analytic continuation, from the real indented line, of both sides of (33) into the entire complex ν-plane is the constant zero. Hence from (33) we have

$$\Phi^-(\nu) = \frac{-I_0(\mu_0 a)}{2\pi\mathrm{i}(\nu-\chi_0)K_+(\chi_0)K_-(\nu)}; \tag{35}$$

which on substitution in (24) and (25) gives,

$$A(\nu) = \frac{-I_0(\mu_0 a)}{2\pi\mathrm{i}(\nu-\chi_0)K_+(\chi_0)K_-(\nu)J(a, 1/\xi, \nu)}, \tag{36}$$

$$B(\nu) = \frac{-I_0(\mu_0 a)H(b, -i/\zeta, \nu)}{2\pi\mathrm{i}(\nu-\chi_0)K_+(\chi_0)K_-(\nu)D(\nu)}, \tag{37}$$

$$C(\nu) = \frac{I_0(\mu_0 a) J(b, -i/\zeta, \nu)}{2\pi\mathrm{i}(\nu - \chi_0) K_+(\chi_0) K_-(\nu) D(\nu)}. \tag{38}$$

Thus, the acoustic field is given everywhere in the duct system by substituting (36)–(38) in (15) and (16), yielding

$$\phi(r, z) = I_0(\mu_0 r)\mathrm{e}^{\mathrm{i}\chi_0 z} - \frac{I_0(\mu_0 a)}{2\pi\mathrm{i}K_+(\chi_0)} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\nu z} J_0(\kappa r)}{(\nu - \chi_{0\smile}) K_-(\nu) J(a, 1/\xi, \nu)} \mathrm{d}\nu, \tag{39}$$

for $(r < a)$;

$$\phi(r, z) = \frac{I_0(\mu_0 a)}{2\pi\mathrm{i}K_+(\chi_0)} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\nu z}[J(b, -\mathrm{i}/\zeta, \nu) H_0^{(1)}(\kappa r) - H(b, -\mathrm{i}/\zeta, \nu) J_0(\kappa r)]}{(\nu - \chi_{0\smile}) K_-(\nu) D(\nu)} \mathrm{d}\nu, \tag{40}$$

for $(a < r < b)$, where the contour of integration is indented below $\nu = \chi_0$.

4 Modal field representation

We can convert the integral representations for the acoustic field, given by (39) and (40), into a series of propagating wave modes. This is achieved by closing the path of integration by a suitable contour and applying Cauchy's residue theorem. We can close the path of integration in (39) or/and (40) by an infinite semi-circle in either $\mathfrak{Im} \geq 0$ or $\mathfrak{Im} \leq 0$ (depending on the sign of z), since both integrands are bounded by $O(\mathrm{e}^{\mathrm{i}\nu z}\nu^{-\frac{3}{2}})$ as $|\nu| \to \infty$ in these regions. We also note that the integrands are even functions of κ and hence there are no branch-point singularities in the entire ν-plane. Thus, an application of Jordan's lemma enables us to close the contour of integration in (39) and (40) by an infinite semi-circle in either $\mathfrak{Im}\nu \geq 0$ or $\mathfrak{Im}\nu \geq 0$ (depending on the sign of z) without affecting the value of the integral. The value of the appropriate integral can then be found as an infinite series of wave modes by summing the residue contributions from the poles of the integrand enclosed by the contour.

4.1 Field in $r < a, z < 0$

Enclosing the contour of integration in (39) by an infinite semi-circle in $\mathfrak{Im}\nu \leq 0$ and summing residues from the only simple poles of the integrand enclosed, that is, the simple zeros of $J(a, 1/\xi, \nu) = 0$, which are given by $\nu = \chi_n (n = 0, 1, 2, \ldots)$, we obtain

$$\phi(r, z) = I_0(\mu_0 r)\mathrm{e}^{\mathrm{i}\chi_0 z} - \frac{I_0(\mu_0 a)}{K_+(\chi_0)} \sum_{m=0}^{\infty} \frac{J_0(\alpha_m r)\mathrm{e}^{-\mathrm{i}\chi_m z}}{(\chi_m + \chi_0) K_+(\chi_m) J'(a, 1/\xi, -\chi_m)}, \tag{41}$$

where $J'(a, 1/\xi, -\chi_m) = \frac{\partial J(a,1/\xi,\nu)}{\partial \nu}|_{\nu=-\chi_m}$.

4.2 Field in $r < a, z > 0$

If we close the contour of integration in (39) by an infinite semi-circle in $\mathfrak{Im}\nu \geq 0$, with $z > 0$, by using Jordan's lemma, and rewrite the integrand of (39) by means of (29), we obtain the equivalent representation for the expression (39) as

$$\phi(r, z) = I_0(\mu_0 r)\mathrm{e}^{\mathrm{i}\chi_0 z} - \frac{I_0(\mu_0 a)a}{4K_+(\chi_0)} \int_{C_+} \frac{\mathrm{e}^{\mathrm{i}\nu z} K_+(\nu) J_0(\kappa r) D(\nu)}{(\nu - \chi_{0\smile}) J(b, -\mathrm{i}/\zeta, \nu)} \mathrm{d}\nu, \tag{42}$$

where C_+ is the closed contour, consisting of the contour of integration in (39) and the infinite circular arc $\mathfrak{Im}\nu \geq 0, |\nu| = R \to \infty$. The only poles enclosed by C_+ are $\nu = \chi_0$ and the roots of $J(b, -\mathrm{i}/\zeta, \nu) = 0$, are $\nu = \xi_m (m = 1, 2, 3, \ldots)$. The residue contribution from the pole $\nu = \chi_0$ *exactly* cancels the first term on the left-hand side of the equality sign in (42), that is, the incident-wave mode.The contribution from the remaining poles gives

$$\phi(r,z) = -\frac{\pi \mathrm{i} a I_0(\mu_0 a)}{2K_+(\chi_0)} \sum_{m=1}^{\infty} \frac{K_+(\xi_m) J_0(\beta_m r) D(\xi_m) \mathrm{e}^{\mathrm{i}\xi_m z}}{(\xi_m - \chi_0) J'(b, -\mathrm{i}/\zeta, \xi_m)}, \tag{43}$$

where $J'(b,-\mathrm{i}/\zeta,\xi_m) = \frac{\partial J(b,-\mathrm{i}/\zeta,\nu)}{\partial \nu}|_{\nu=\xi_m}$, and where $\nu = \xi_m$ are the zeros of $J(b,-\mathrm{i}/\zeta,\nu) = 0$, and $\beta_m = \sqrt{k^2 - \xi_m^2}$ $(m = 1, 2, 3 \ldots)$.

4.3 Field in $a < r < b, z < 0$

In the expression (40) the functions $K_-(\nu)$ and $(\nu - \chi_0)$ do not vanish in the region $\mathfrak{Im}\nu \leq 0$, $\nu \neq \chi_0$, and thus the only singularities of the integrand in this region are the poles corresponding to the zeros of $D(\nu) = 0$, that is, $\nu = -\eta_m (m = 1, 2, 3, \ldots)$. If we close the contour of integration in (40) by an infinite semi-circular arc in $\mathfrak{Im}\nu \leq 0$, with $z < 0$, we get, on summing the residue contributions,

$$\phi(r,z) = -\frac{I_0(\mu_0 a)}{K_+(\chi_0)} \sum_{m=1}^{\infty} \frac{[J(b,-\mathrm{i}/\zeta,-\eta_m) H_0^{(1)}(\delta_m r) - H(b,-\mathrm{i}/\zeta,-\eta_m) J_0(\delta_m r)]}{(\eta_m + \chi_0) K_+(\eta_m) D'(-\eta_m)} \mathrm{e}^{-\mathrm{i}\eta_m z}, \tag{44}$$

where $D'(-\eta_m) = \frac{\partial D(\nu)}{\partial \nu}|_{\nu=-\eta_m}$ and where $\delta_m = \sqrt{k^2 - \eta_m^2}$ are the zeros of (13).

4.4 Field in $a < r < b, z > 0$

If we close the contour of integration in (40) by an infinite semi-circular arc in $\mathfrak{Im}\nu \geq 0$, by applying Jordan's lemma, and rewrite the integrand by means of (13), we have the equivalent representation for (40) given by

$$\phi(r,z) = \frac{a I_0(\mu_0 a)}{4K_+(\chi_0)} \int_{C_+} \frac{\mathrm{e}^{\mathrm{i}\nu z} J(a, 1/\xi, \nu) K_+(\nu)}{(\nu - \chi_{0\smile}) J(b,-\mathrm{i}/\zeta,\nu)} \left[J(b,-\mathrm{i}/\zeta,\nu) H_0^{(1)}(\kappa r) - H(b,-\mathrm{i}/\zeta,\nu) J_0(\kappa r) \right] \mathrm{d}\nu, \tag{45}$$

where C_+ is the same closed contour as that in (42). There is no residue contribution from the apparent pole $\nu = \chi_0$ because this is a removable singularity, cancelled by the zero $\nu = \chi_0$ of $J(a, 1/\xi, \nu) = 0$. Thus, the only residue contribution arises from the zeros of $J(b,-\mathrm{i}/\zeta,\nu) = 0$, that is, $\nu = \xi_m (m = 1, 2, 3, \ldots)$. Thus

$$\phi(r,z) = -\frac{\pi \mathrm{i} a I_0(\mu_0 a)}{2K_+(\chi_0)} \sum_{m=1}^{\infty} \frac{J(a, 1/\xi, \xi_m) K_+(\xi_m) H(b,-\mathrm{i}/\zeta,\xi_m) J_0(\beta_m r) \mathrm{e}^{\mathrm{i}\xi_m z}}{(\xi_m - \chi_0) J'(b,-\mathrm{i}/\zeta,\xi_m)}, \tag{46}$$

which is the same as (43), since $D(\xi_m) = J(a, 1/\xi, \xi_m) H(b,-\mathrm{i}/\zeta,\xi_m)$, and this is what we would expect from the physics of the problem.

5 Dominant behavior of field

For the propagation of the dominant surface-wave mode the resultant dominant behavior of the acoustic field in the various regions is given by

$$\phi(r,z) = I_0(\mu_0 r) \mathrm{e}^{\mathrm{i}\chi_0 z} + I_0(\mu_0 r) \mathrm{e}^{-\mathrm{i}\chi_0 z} \left(\frac{\mu_0^2 \xi^2}{2a\chi_0^2 (\mu_0^2 \xi^2 - k^2)(K_+(\chi_0))^2} \right) + O(\mathrm{e}^{-\mathrm{i}\chi_1 z}), \tag{47}$$

for $(0 \leq r \leq a, -\infty < z < 0)$.

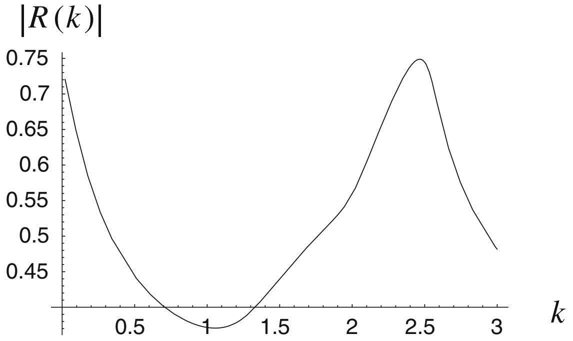

Fig. 2 Graph of the modulus of the reflection coefficient $|R(k)|$, as a function of k for $a = 1, b = 1.5, \xi = -1, \zeta = 10 + \iota$; $|R(0)| = 1.0, k_0 = 3.1128$

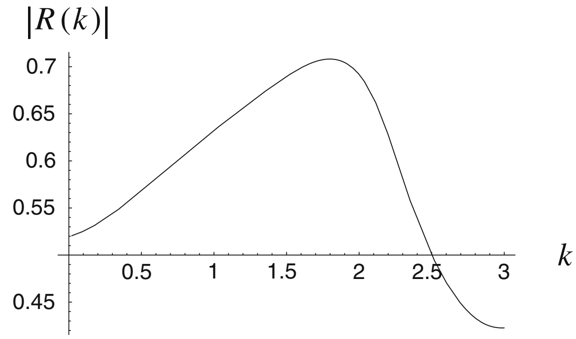

Fig. 3 Graph of the modulus of the reflection coefficient $|R(k)|$, as a function of k for $a = 1, b = 1.5, \xi = -1, \zeta = 1 + \iota$; $|R(0)| = 1.0, k_0 = 3.1128$

$$\phi(r, z) = -\frac{\pi \mathrm{i} a I_0(\mu_0 a)}{2bK_+(\chi_0)} \frac{\beta_1^2 \zeta^2 K_+(\xi_1) J(a, 1/\xi, \xi_1) H(b, -\mathrm{i}/\zeta, \xi_1) J_0(\beta_1 r) \mathrm{e}^{\mathrm{i}\xi_1 z}}{\xi_1(\xi_1 - \chi_0)(k^2 - \beta_1^2 \zeta^2) J_0(\beta_1 b)} + O(\mathrm{e}^{\mathrm{i}\xi_2 z}), \tag{48}$$

for $(0 \le r \le b, 0 < z < \infty)$.

$$\phi(r, z) = \frac{I_0(\mu_0 a)}{K_+(\chi_0)} \frac{[J(b, -\mathrm{i}/\zeta, -\eta_1) H_0^{(1)}(\delta_1 r) - H(b, -\mathrm{i}/\zeta, -\eta_1) J_0(\delta_1 r)]}{(\eta_1 + \chi_0) K_+(\eta_1) D'(-\eta_1)} \mathrm{e}^{-\mathrm{i}\eta_1 z} + O(\mathrm{e}^{-\mathrm{i}\eta_2 z}), \tag{49}$$

for $(a \le r \le b, -\infty < z < 0)$, where

$$\begin{aligned} D'(\nu) = {} & J'(a, 1/\xi, \nu) H(b, -\mathrm{i}/\zeta, \nu) + J(a, 1/\xi, \nu) H'(b, -\mathrm{i}/\zeta, \nu) \\ & - J'(b, -\mathrm{i}/\zeta, \nu) H(a, 1/\xi, \nu) - J(b, -\mathrm{i}/\zeta, \nu) H'(a, 1/\xi, \nu), \end{aligned}$$

with

$$J'(\alpha, \beta, \nu) = \alpha\nu(\kappa J_0(\kappa a) - k\beta J_0'(\kappa a))/\kappa,$$

$$H'(\alpha, \beta, \nu) = \alpha\nu(\kappa H_0^{(1)}(\kappa a) - k\beta H_0^{(1)\prime}(\kappa a))/\kappa.$$

From the results (47)–(49) it is an easy matter to obtain the reflection and transmission coefficients for the dominant surface-wave mode propagating in the various regions. In particular, the reflection coefficient R (=incident surface-wave mode/reflected surface-wave mode) calculated at $z = 0$ of the incident wave reflected back into the duct $(0 \le r \le a, -\infty < z < 0)$, is given by

$$R(k) = \frac{\mu_0^2 \xi^2}{2a\chi_0^2(\mu_0^2\xi^2 - k^2)(K_+(\chi_0))^2} = -|R|\mathrm{e}^{2\mathrm{i}kl}. \tag{50}$$

Hence the modulus of the reflection coefficient is given by using the results of Appendix D by the compact expression

$$|R(k)| = \left|\frac{\pi a}{2} I_0(\mu_0 a) H(a, 1/\xi, \chi_0)\right| \mathrm{e}^{-\frac{2\chi_0}{\pi} \mathrm{P} \int_0^\infty \frac{arg H(t)}{t^2 - \chi_0^2} \mathrm{d}t}, \tag{51}$$

where the P in front of the integral sign denotes the principal value of the integral, and $H(t) = (t^2 - \chi_0^2)K(t)$. The last expression for$|R(k)|$ is used to produce the graphs shown in Figs. 2 and 3 above. The value of $|R|$ can be calculated for $k = 0$ by using the static method often used in acoustic waveguide theory. This assumes that the fundamental mode propagates in all the duct regions with all duct surfaces being rigid. This calculation gives $R = (a^2 - b^2)/b^2$, which for $a = 1, b = 1.5$ gives $R = 0.555556$. This does not agree numerically with the above expression (51) when k is put equal to zero, i.e., $|R(0)| = 1.0$. However, the limit is not uniform at $k = 0$. It will be noticed from the graphs that it is not the same as $\lim_{k \to 0}|R(k)|$. It will also be noticed that the variation of $|R(k)|$ with ζ is significant enough to detect variations in the impedance lining.

6 Conclusions

We have solved exactly a new boundary-value problem involving surface-wave propagation in a lined cylindrical duct. We have been able to numerically evaluate the reflection coefficient for the dominant surface-wave mode, which is of practical importance in applications, and which involves complicated split functions (that arise from the Wiener–Hopf technique). This is achieved by numerically evaluating the split functions defined in terms of suitable Cauchy integrals, rather than the usual method of infinite products; see the references cited in [14]. The numerical evaluation of the infinite product would normally be a non-trivial matter because of the infinite number of complex factors. However, the use of Mathematica to numerically evaluate suitable Cauchy integrals very effectively overcomes this problem. The simplistic static approximation used by engineers to calculate reflection and by association the transmission coefficients does not seem to be accurate enough when dealing with lined ducts. The significant variation of the reflection coefficient for changes in the impedance lining does offer an instrument for the detection of a change in the properties of the wall lining. The numerical evaluation of the transmission coefficients in the other duct regions offers no great analytical difficulties, and can be achieved by the methods used here in evaluating the reflection coefficient. A full numerical evaluation of the fields in the various regions of the duct system and their dependance on the large number of physical parameters needs to be considered in a future publication. Although we have assumed that $\mathfrak{Re}\zeta > 0$ the same method used here can *mutatis—mutandis*, be used to solve the situation where $\mathfrak{Re}\zeta = 0, \mathfrak{Im}\zeta \neq 0$. In this case we would have a surface-wave transformer, which would convert a surface-wave generated in $z < 0$ to a new surface wave in $z > 0$. In principle, no substantial difficulties would prevent us from obtaining exact closed-form solutions for other incident-wave-mode situations with this waveguide system. For example, by reversing the sign of ξ, we could consider the incident-wave mode in the annular region $a < r < b, z < 0$. We have also extended the usual Sturm–Liouville method in the appendices to give useful information on the disposition of the poles and zeros of the complicated special-function eigenvalue equations that arise from third-type boundary conditions with complex coefficients. We note that by letting $\xi \to \infty$ we obtain the solution to the problem of the radiation from a rigid semi-infinite duct into an infinite lined duct that was given by [14]. Finally, we intend to deal with the important extensions of this work for the electromagnetic communication in subsurface tunnels in a future publication.

Appendix A: Normal modes in infinite duct regions

Here we shall derive the permissible normal wave modes $\psi(r, z)$ that can propagate in the various duct regions. For this purpose we need only consider solutions of the wave equation $(\nabla^2 + k^2)\psi = 0$, in the infinite region $-\infty < z < \infty$ for various ranges of r. We shall also assume here that k and ξ are real, whereas ζ can be complex.

A.1 Normal modes in $0 < r < a$

Here we have to satisfy the boundary condition:

$$\frac{\partial \psi}{\partial r} + \frac{k}{\xi}\psi = 0, \quad (r = a, -\infty < z < \infty).$$

By separation of variables, it is not difficult to show that the only permissible modes are given by

$$\psi(r, z) = \mathrm{e}^{\pm \mathrm{i}\chi_n z} J_0(\alpha_n r), \quad (n = 0, 1, 2, \ldots), \tag{52}$$

where $\chi_n = (k^2 - \alpha_n^2)^{\frac{1}{2}}$ and α_n are the real roots of the equation $\alpha_n J_0'(\alpha_n a) + (k/\xi) J_0(\alpha_n a) = 0$. In the expression (52), when χ_n is real and positive, the upper sign represents an outgoing wave as $z \to \infty$, whereas the lower sign represents an outgoing wave at $z \to -\infty$. From the way the square root has been defined, χ_n can only be real positive or purely imaginary positive. In the situation where χ_n is purely positive imaginary, the upper(lower) sign in (52) represents bounded evanescent waves as $z \to \infty(-\infty)$. It should be noticed that in carrying out the separation of variables the differential equation and boundary conditions for the r variable problem reduces to a classical Sturm–Liouville boundary problem. We can use this theory to assert that there are an infinite number of real eigenvalues α_n such that there exists a finite lower bound α_0^2. That is, we can order the eigenvalues $\alpha_0^2 < \alpha_1^2 < \alpha_2^2 < \alpha_3^2 < \cdots$ where α_0^2 may be negative, and hence α_0 purely imaginary. The incident field $\psi_0(r, z)$ will be assumed to correspond to the lowest possible mode propagating as a surface wave towards $z \to \infty$. That is

$$\psi_0(r, z) = J_0(\alpha_0 r)\mathrm{e}^{\mathrm{i}\chi_0 z},$$

where α_0, is the solution of the equation

$$\alpha_0 J_0'(\alpha_0 a) + (k/\xi) J_0(\alpha_0 a) = 0,$$

such that α_0^2 is the smallest value of all solutions of the equation $\alpha_n J_0'(\alpha_n a) + (k/\xi) J_0(\alpha_n a) = 0$, $(n = 0, 1, 2, 3...)$.

We now investigate the roots of the equation $\alpha_0 J_1(\alpha_n a) - (k/\xi) J_0(\alpha_n a) = 0$, corresponding to the dominant propagating surface-wave mode in the semi-infinite region $0 \le r \le a$, $z < 0$. For propagating modes we must have $\chi_n = \sqrt{k^2 - \alpha_n^2} > 0$, which means that these modes can only occur:

(i) when α_n is purely positive imaginary, say $\alpha_n = \mathrm{i}\mu_n$, $\mu_n > 0$, so that $\chi_n = \sqrt{k^2 + \mu_n^2}$ and the propagating mode becomes $J_0(\mathrm{i}\mu_n r)\mathrm{e}^{\mathrm{i}\chi_n z} = I_0(\mu_n r)\mathrm{e}^{\mathrm{i}\chi_n z}$. This corresponds to a surface wave whose amplitude decays from $I_0(\mu_n a) > 1$ at $r = a$ to unity at $r = 0$.

(ii) when α_n real with$-k < \alpha_n < k$ so that $\chi_n = \sqrt{k^2 - \alpha_n^2} > 0$. We shall only consider the case (i), that is α purely positive imaginary which corresponds to a surface wave propagating. Thus with $\alpha = \mathrm{i}\mu$, $\mu > 0$, the modal equation (8) can be written as,

$$\frac{\mu I_1(\mu a)}{I_0(\mu a)} = -\frac{k}{\xi} > 0. \tag{53}$$

We also have the representation:

$$\mu \frac{I_1(\mu a)}{I_0(\mu a)} = 2\mu^2 a \sum_{n=1}^{\infty} \frac{1}{(\mu^2 a^2 + \gamma_{\nu,0}^2)} > 0,$$

for $\mu > 0$. And hence

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left\{\mu \frac{I_1(\mu a)}{I_0(\mu a)}\right\} = 4\mu a \sum_{n=1}^{\infty} \frac{\gamma_{\nu,0})^2}{(\mu^2 a^2 + \gamma_{\nu,0}^2)^2} > 0.$$

Thus $\mu \frac{I_1(\mu a)}{I_0(\mu a)}$ is a monotonic increasing function of μ with $\lim_{\mu \to 0} \frac{\mu I_1(\mu a)}{I_0(\mu a)} = 0$. Thus Eq. (8) only has one root for $-\frac{k}{\xi} > 0$. We shall now prove that the equation $F(z) = AzJ_0'(z) + BJ_0(z) = 0$ has an infinite number of real roots for real finite A and B. For $A = B = 0$ the result is obvious. If $B = 0$, $A \neq 0$ then $F(z) = 0$ has an infinite number of real zeros corresponding to the roots of $zJ_0'(z) = 0$. Similarly, for $B \neq 0$, $A = 0$, $F(z) = 0$, has an infinite number of real zeros corresponding to $J_0(z) = 0$. We need now only consider A and B finite. Here we let $0 < j_1 < j_2 < j_3 < j_4 < \cdots$ be the zeros of $J_0(z)$. Then since $J_0(z)$ is positive in the interval $(0, j_1)$, negative in the interval (j_1, j_2), positive in (j_2, j_3), etc..., we have

$$J_0'(j_1) < 0,\ J_0'(j_2) > 0,\ J_0'(j_3) < 0,\ J_0'(j_4) > 0,\ J_0'(j_5) < 0 \cdots$$

Hence $F(z)$ alternates in sign at the points $j_1, j_2, j_3, j_4, \ldots$ and therefore vanishes somewhere between each of them. To ensure that only the surface wave propagates in the semi-infinite cylinder $r < a$, $z < 0$ and no waves of type (ii) we shall require $\alpha_1^2 > k^2$. The appropriate range for k can be found from the smallest positive solution of the equation $\alpha J_0'(a\alpha) + k/\xi\, J_0(a\alpha) = 0$ when $\alpha = k$. Thus we solve $\xi J_0'(ka) + J_0(ka) = 0$ for the smallest $k > 0$. Let us denote this root by k_0. It is not difficult to show that for all $\xi < 0$, then $2.4048 < k_0 a < 5.5200$.

A.2 Normal modes in $0 < r < b$

Here we have to satisfy the boundary condition:

$$\frac{\partial \psi}{\partial r} - \frac{\mathrm{i}k}{\zeta}\psi = 0, \quad (r = b, -\infty < z < \infty).$$

The only permissible modes of propagation are given by the method of separation of variables as:

$$\psi(r, z) = \mathrm{e}^{\pm \mathrm{i}\xi_n z} J_0(\beta_n r), \quad (n = 1, 2, 3, \ldots), \tag{54}$$

where $\beta_n = (k^2 - \xi_n^2)^{\frac{1}{2}}$ and β_n are the complex roots of the equation

$$\beta_n J_1(\beta_n b) + (\mathrm{i}k/\zeta) J_0(\beta_n b) = 0, \quad (n = 1, 2, 3, \ldots).$$

It is shown in the Appendix B that for $\mathfrak{Re}\zeta > 0$

$$\mathfrak{Re}\,\xi_n \mathfrak{Im}\xi_n > 0.$$

Thus, the upper(lower) sign of the expression (54) represent outgoing bounded waves as $z \to \infty(-\infty)$.

A.3 Normal modes in $a < r < b$

Here we have to satisfy the two boundary conditions

$$\frac{\partial \psi}{\partial r} + \frac{k}{\xi}\psi = 0, \quad (r = a, -\infty < z < \infty),$$

$$\frac{\partial \psi}{\partial r} - \frac{\mathrm{i}k}{\zeta}\psi = 0, \quad (r = b, -\infty < z < \infty).$$

By an application of separation of variables, the possible modes are given by assuming a solution of the wave equation of the form

$$\psi(r, z) = \mathrm{e}^{\pm \mathrm{i}\eta_n z}(A J_0(\delta_n r) + B H_0^{(1)}(\delta_n r)),$$

where $\delta_n = (k^2 - \eta_n^2)^{\frac{1}{2}}$. On substituting the above expression in the boundary condition at $r = a$ we have

$$A\left(\delta_n J_0'(\delta_n a) + \frac{k}{\xi} J_0(\delta_n a)\right) + B\left(H_0^{(1)\prime}(\delta_n a) + \frac{k}{\xi} H_0^{(1)}(\delta_n a)\right) = 0.$$

A solution of the above equation is given by choosing

$$A = \left(H_0^{(1)\prime}(\delta_n a) + \frac{k}{\xi} H_0^{(1)}(\delta_n a)\right) C,$$

$$B = -\left(\delta_n J_0'(\delta_n a) + \frac{k}{\xi} J_0(\delta_n a)\right) C.$$

Thus

$$\psi(r, z) = \mathrm{e}^{\pm \mathrm{i}\eta_n z} C\left[(H_0^{(1)\prime}(\delta_n a) + \frac{k}{\xi} H_0^{(1)}(\delta_n a)) J_0(\delta_n r) - \left(\delta_n J_0'(\delta_n a) + \frac{k}{\xi} J_0(\delta_n a)\right) H_0^{(1)}(\delta_n r)\right].$$

Substituting the latter expression in the boundary condition on $r = b$ will give non-zero values of C, and hence non trivial solutions of the boundary-value problem if

$$\left(\delta_n H_0^{(1)\prime}(\delta_n a) + \frac{k}{\xi} H_0^{(1)}(\delta_n a)\right)\left(\delta_n J_0'(\delta_n b) - \frac{\mathrm{i}k}{\zeta} J_0(\delta_n b)\right) - (\delta_n H_0^{(1)\prime}(\delta_n b) + \frac{k}{\xi} H_0^{(1)}(\delta_n b))(\delta_n J_0'(\delta_n a)$$
$$- \frac{\mathrm{i}k}{\zeta} J_0(\delta_n a)) = 0.$$

It is shown in the appendix B that for real ξ and $\mathfrak{Re}\zeta > 0$ then

$$\mathfrak{Im}\eta_n \mathfrak{Re}\,\eta_n > 0.$$

Appendix B: Location of the complex eigenvalues

B.1 Position of complex wavenumbers χ_n, and ξ_n

Here we shall extend the Sturm–Liouville method to give information on the nature of the real and imaginary parts of complex eigenvalues as a function of a complex parameter Z that appears in the third-type boundary conditions for a cylindrical region. Upon application of the method of separation of variables to the wave equation in cylindrical coordinates (r, z), $(\nabla^2 + k^2)\psi = 0$, the substitution of $\psi(r, z) = \phi(r)\mathrm{e}^{\mathrm{i}\xi z}$ results in the radial eigenvalue value problem:

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\phi'(r)) - \frac{\xi^2\phi(r)}{r} = -k^2 r\phi(r), \quad (0 < r < b);$$

$$\phi'(b) - \mathrm{i}kZ\phi(b) = 0, \quad \lim_{r\to 0}\phi(r) = A, \quad \lim_{r\to 0}\phi'(r) = 0,$$

where Z is complex, and A is a bounded constant. The boundary condition is appropriate for a bounded solution that satisfies the absorbing boundary condition on the duct wall. If we conjugate the last two equations, we get the equivalent eigenvalue equations for the conjugate function $\overline{\phi}$

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\overline{\phi}'(r)) - \frac{\overline{\xi}^2\overline{\phi}(r)}{r} = -k^2 r\overline{\phi}(r),$$

$$\overline{\phi}'(b) + \mathrm{i}k\overline{Z\phi}(b) = 0, \quad \lim_{r\to 0}\overline{\phi}(0) = \overline{A}, \quad \lim_{r\to 0}\overline{\phi}'(r) = 0.$$

Now multiply the first differential equation problem for ϕ across by $\overline{\phi}$ and the second differential equation problem for $\overline{\phi}$ by ϕ. This gives the set of problems:

$$\overline{\phi(r)}\frac{\mathrm{d}}{\mathrm{d}r}(r\phi'(r)) - \frac{\xi^2|\phi(r)|^2}{r} = -k^2 r|\phi(r)|^2;$$

$$\phi(r)\frac{\mathrm{d}}{\mathrm{d}r}(r\overline{\phi}'(r)) - \overline{\xi}^2\frac{|\phi(r)|^2}{r} = -k^2 r|\phi(r)|^2.$$

By subtracting the last two differential equations one from the other, we get

$$\phi(r)\frac{\mathrm{d}}{\mathrm{d}r}(r\overline{\phi}'(r)) - \overline{\phi(r)}\frac{\mathrm{d}}{\mathrm{d}r}(r\phi'(r)) - (\overline{\xi}^2 - \xi^2)\frac{|\phi(r)|^2}{r} = 0,$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}r}[r(\overline{\phi}'(r)\phi(r) - \overline{\phi}(r)\phi'(r))] = (\overline{\xi}^2 - \xi^2)\frac{|\phi(r)|^2}{r}.$$

Integrating both sides of the last expression with respects to r from $r \to 0^+$ to $r = b$, we have

$$(\overline{\xi}^2 - \xi^2)\int_{0^+}^{b}\frac{|\phi(r)|^2}{r}\mathrm{d}r = [r(\overline{\phi}'(r)\phi(r) - \overline{\phi}(r)\phi'(r))]_{0^+}^{b}.$$

From the boundary conditions we have

$$|r(\overline{\phi}'(r)\phi(r) - \overline{\phi}(r)\phi'(r))|_{0^+} = 0,$$

and

$$|r(\overline{\phi}'(r)\phi(r) - \overline{\phi}(r)\phi'(r))|^{b} = \mathrm{i}k(Z + \overline{Z})|\phi(b)|^2,$$

so that

$$(\overline{\xi}^2 - \xi^2)\int_{0^+}^{b}\frac{|\phi(r)|^2}{r}\mathrm{d}r = -\mathrm{i}k(Z + \overline{Z})|\phi(b)|^2,$$

or

$$-2\mathrm{i}\mathfrak{Re}\,\xi\mathfrak{Im}\xi\int_{0+}^{b}\frac{|\phi(r)|^2}{r}dr=-2\mathrm{i}k\mathfrak{Re}\;Z|\phi(b)|^2.$$

For consistency of sign on both sides of the last expression, if $\mathfrak{Re}\;Z>0$, we must have

$\mathfrak{Re}\xi\mathfrak{Im}\xi>0.$

Note that if $\mathfrak{Re}\;Z=0$ then $\mathfrak{Re}\xi\mathfrak{Im}\xi=0$ so that the eigenvalues ξ must lie on the real or imaginary axes; this is the situation for the modes χ_n that propagate in the cylindrical region $0<r<a$.

B.2 Position of the complex wavenumbers η_n

Here we shall give information on the nature of the real and imaginary parts of complex eigenvalues as a function of a complex parameter ζ and the real parameter ξ that appear in the third-type boundary conditions in an annular cylindrical duct region $0<a<r<b$. Upon application of the method of separation of variables to the wave equation in cylindrical coordinates, the substitution of $\psi(r,z)=\phi(r)\mathrm{e}^{\mathrm{i}\eta z}$ results in the radial eigenvalue value problem:

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\phi'(r))-\frac{\eta^2\phi(r)}{r}=-k^2r\phi(r),\quad (a<r<b);$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\phi(a)+\frac{k}{\xi}\phi(a)=0,\quad \frac{\mathrm{d}}{\mathrm{d}r}\phi(b)-\frac{\mathrm{i}k}{\zeta}\phi(b)=0.$$

By conjugating the above boundary-value problem, we get

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\overline{\phi}'(r))-\frac{\overline{\eta}^2\overline{\phi}(r)}{r}=-k^2r\overline{\phi}(r);$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\overline{\phi}(a)+\frac{k}{\xi}\overline{\phi}(a)=0,\quad \frac{\mathrm{d}}{\mathrm{d}r}\overline{\phi}(b)+\frac{\mathrm{i}k}{\overline{\zeta}}\overline{\phi}(b)=0.$$

Multiplying across the differential equation for ϕ by $\overline{\phi}$, and the differential equation for $\overline{\phi}$ by ϕ and subtracting the resulting equations one from the other, we have

$$\phi(r)\frac{\mathrm{d}}{\mathrm{d}r}(r\overline{\phi}'(r))-\overline{\phi(r)}\frac{\mathrm{d}}{\mathrm{d}r}(r\phi'(r))-(\overline{\eta}^2-\eta^2)\frac{|\phi(r)|^2}{r}=0,$$

By integrating across this equation from $r=a$ to $r=b$, we get

$$(\overline{\eta}^2-\eta^2)\int_a^b\frac{|\phi(r)|^2}{r}\mathrm{d}r=[r(\overline{\phi}'(r)\phi(r)-\overline{\phi}(r)\phi'(r))]_a^b.$$

From the boundary condition on $r=a$ for ϕ and $\overline{\phi}$ we have on multiplying across by $\overline{\phi}$ and ϕ, respectively,

$$\overline{\phi}(a)\frac{\mathrm{d}}{\mathrm{d}r}\phi(a)+\frac{k}{\xi}|\phi(a)|^2=0,\quad \phi(a)\frac{\mathrm{d}}{\mathrm{d}r}\overline{\phi}(a)+\frac{k}{\xi}|\phi(a)|^2=0,$$

from which we get the result

$$\overline{\phi}(a)\frac{\mathrm{d}}{\mathrm{d}r}\phi(a)-\phi(a)\frac{\mathrm{d}}{\mathrm{d}r}\overline{\phi}(a)=0.$$

Again from the boundary condition on $r=b$ for ϕ and $\overline{\phi}$ we have on multiplying across by $\overline{\phi}$ and ϕ, respectively,

$$\overline{\phi}(b)\frac{\mathrm{d}}{\mathrm{d}r}\phi(b)-\frac{\mathrm{i}k}{\zeta}|\phi(b)|^2=0,\quad \phi(b)\frac{\mathrm{d}}{\mathrm{d}r}\overline{\phi}(b)+\frac{\mathrm{i}k}{\overline{\zeta}}|\phi(b)|^2=0,$$

from which we get the result

$$\overline{\phi}(b)\frac{\mathrm{d}}{\mathrm{d}r}\phi(b)-\phi(b)\frac{\mathrm{d}}{\mathrm{d}r}\overline{\phi}(b)=\frac{\mathrm{i}k(\overline{\zeta}+\zeta)}{|\zeta|^2}|\phi(b)|^2.$$

By using these results in the last integral expression we get

$$(\overline{\eta}^2-\eta^2)\int_a^b \frac{|\phi(r)|^2}{r}\mathrm{d}r=\frac{\mathrm{i}k(\overline{\zeta}+\zeta)}{|\zeta|^2}|\phi(b)|^2,$$

or

$$\mathfrak{Re}\eta\mathfrak{Im}\eta\int_a^b \frac{|\phi(r)|^2}{r}\mathrm{d}r=\frac{k|\phi(b)|^2}{|\zeta|^2}\mathfrak{Re}\,\zeta.$$

If $\mathfrak{Re}\zeta>0$ then necessarily $\mathfrak{Re}\eta\mathfrak{Im}\eta>0$.

Appendix C: Edge condition

On substituting (15) in the expression $\frac{\partial\phi}{\partial r}+\frac{k}{\xi}\phi$ with $r=a$, we obtain

$$\frac{\partial\phi}{\partial r}+\frac{k}{\xi}\phi=\left[\mu_0 I_0'(\mu_0)+\frac{k}{\xi}I_0(\mu_0 a)\right]\mathrm{e}^{\mathrm{i}\chi_0 z}+\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\nu z}A(\nu)\left[\kappa J_0'(\kappa a)+\frac{k}{\xi}J_0(\kappa a)\right]\mathrm{d}\nu.$$

Now $\mu_0 I_0'(\mu_0)+\frac{k}{\xi}I_0(\mu_0 a)=0$, so that

$$\frac{\partial\phi}{\partial r}+\frac{k}{\xi}\phi=\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\nu z}A(\nu)\left[\kappa J_0'(\kappa a)+\frac{k}{\xi}J_0(\kappa a)\right]\mathrm{d}\nu,$$

but $\frac{\partial\phi}{\partial r}+\frac{k}{\xi}\phi=0$ for $z<0$ so that

$$A(\nu)\left[\kappa J_0'(\kappa a)+\frac{k}{\xi}J_0(\kappa a)\right]=\varphi_-(\nu),$$

where the function $\varphi_-(\nu)$ is holomorphic in $\mathfrak{Im}\nu<0$. Thus, from the edge condition (14) as $z\to 0$,

$$\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\nu z}\varphi_-(\nu)\mathrm{d}\nu=O(z^{-\frac{1}{2}})+\frac{k}{\xi}O(1).$$

Hence by well-known Fourier asymptotics, as $|\nu|\to\infty$, we have $\varphi_-(\nu)=O(|\nu|^{-\frac{1}{2}})$ so that $A(\nu)[O(|\nu|^{\frac{1}{2}}\mathrm{e}^{|\nu|a})+\frac{k}{\xi}O(|\nu|^{-\frac{1}{2}}\mathrm{e}^{|\nu|a})]=O(\nu^{-\frac{1}{2}})$, which implies that

$$A(\nu)=O(|\nu|^{-1}\mathrm{e}^{-|\nu|a}),\quad(|\nu|\to\infty).$$

By the same method we can also show that

$$\frac{\mathrm{e}^{a|\nu|}}{\sqrt{\pi}}B(\nu)+\sqrt{\pi}\mathrm{e}^{-a|\nu|}C(\nu)=O(|\nu|^{-1}),\quad(|\nu|\to\infty).$$

Appendix D: Explicit expression for $K_+(\nu)$ and $|K_+(\chi_0)|$

The function

$$K(\nu)=\left(\frac{2}{\mathrm{i}\pi a}\right)\frac{J(b,-\mathrm{i}/\zeta,\nu)}{J(a,1/\xi,\nu)D(\nu)},$$

is even in ν and κ. We now use the method, given in [15, p 17] to express $K(\nu)=K_+(\nu)K_-(\nu)$ with

$$K_+(\nu)=(K^*(\nu))^{\frac{1}{2}}\mathrm{e}^{\frac{1}{2}g(\nu)},\quad K_-(\nu)=(K^*(\nu))^{\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}g(\nu)},$$

where $K^*(\nu)=1$ for $\mathfrak{Im}\nu\neq 0$, and $K^*(\nu)=K(\nu)$ for $\mathfrak{Im}\nu=0$; and

$$g(\nu)=\frac{1}{2\pi\mathrm{i}}\int_{-i\epsilon-\infty}^{-i\epsilon+\infty}\frac{\log K(t)dt}{t-\nu}+\frac{1}{2\pi\mathrm{i}}\int_{i\epsilon-\infty}^{i\epsilon+\infty}\frac{\log K(t)dt}{t-\nu},$$
$$=\frac{1}{\pi\mathrm{i}}\mathrm{P}\int_{-\infty}^{\infty}\frac{\log K(t)dt}{t-\nu}=\frac{1}{\pi\mathrm{i}}\mathrm{P}\int_{-\infty}^{\infty}\frac{(\log|K(t)|+\mathrm{i}\arg(K(t)))dt}{t-\nu},$$

where the P in front of the integral sign denotes the principal value integral. The above representations for the functions $K_{\pm}(\nu)$ are satisfactory for analytic purposes. For example, see [15, pp. 41–42], the fact that $|K(\nu)| \sim 2|\nu|^{-1}$ implies that $|K_{\pm}(\nu)| \sim \sqrt{2}|\nu|^{-\frac{1}{2}}$ as $|\nu| \to \pm\infty$. However, if zeros or poles of $K(\nu)$ lie on the path of integration the numerical evaluation becomes delicate. Here, for the numerical calculation of the reflection coefficient $|R|$, we will need to calculate $K_+(\chi_0)$ and $J(a, 1/\xi, \chi_0) = 0$. We shall therefore modify the method to obtain a more suitable form for the numerical evaluation of $K_+(\chi_0)$. To this end we factor out the poles of $K(\nu)$ by defining a new function $H(\nu) = (\nu^2 - \chi_0^2)K(\nu)$. Then $H(\nu)$ is even in ν and $H(t) \sim 2|t|$ as $t \to \pm\infty$. Then $H(\nu)$ gives no problems, so we can write

$$H_+(\nu) = (H(\nu))^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}h(\nu)}, \quad H_-(\nu) = (H(\nu))^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}h(\nu)},$$

$$h(\nu) = \frac{2\nu}{\pi \mathrm{i}} \mathrm{P} \int_0^{\infty} \frac{\log H(t)\mathrm{d}t}{t^2 - \nu^2}.$$

The existence of the above principal-value integral is assured because of the evenness of $H(t)$ and the fact that $\log H(t) \sim \log |t|$ as $t \to \pm\infty$. Also $H(t) \neq 0$ along the contour of integration. Thus

$$K_+(\nu) = \frac{H_+(\nu)}{(\nu + \chi_0)},$$

$$K_+(\nu) = \sqrt{\frac{\nu - \chi_0}{\nu + \chi_0} K(\nu)}\, \mathrm{e}^{\frac{1}{2\pi \mathrm{i}} \mathrm{P} \int_{-\infty}^{\infty} \frac{\log[(t^2 - \chi_0^2)K(t)]\mathrm{d}t}{t - \nu}}.$$

Hence

$$|K_+(\nu)|^2 = \left| K(\nu) \frac{(\nu - \chi_0)}{(\nu + \chi_0)} \mathrm{e}^{h(\nu)} \right|.$$

In the limit as $\nu \to \chi_0$, $\kappa \to \mathrm{i}\mu_0$ it can be shown that

$$|K_+(\chi_0)|^2 = \left| \left(\frac{J(b, -\mathrm{i}/\zeta, \chi_0)}{\mathrm{i}\pi a \chi_0 D(\chi_0) J'(a, 1/\xi, \chi_0)} \right) \mathrm{e}^{h(\chi_0)} \right|.$$

Since

$$D(\chi_0) = -J(b, -\mathrm{i}/\zeta, \chi_0) H(a, 1/\xi, \chi_0),$$

and

$$J'(a, 1/\xi, \chi_0) = \frac{\chi_0 a}{\mu_0^2 \xi^2}((\mu_0 a)^2 - k^2) I_0(\mu_0 a);$$

we have the final result

$$|K_+(\chi_0)|^2 = \left| \frac{\mu_0^2 \xi^2 \mathrm{e}^{h(\chi_0)}}{\pi a^2 \chi_0^2 H(a, 1/\xi, \chi_0)((\mu_0 a)^2 - k^2) I_0(\mu_0 a)} \right|.$$

References

1. Mahmoud SF (1991) Electromagnetic waveguides theory and practice. Peter Peregrinus Ltd
2. Barlow HM, Brown J (1962) Radio surface waves. Oxford University press
3. Hecht J (1990) Understanding Fiber optics. Prentice Hall
4. Snyder AW, Love JD (1983) Optical waveguide theory. Chapman and Hall
5. Morse PM (1939) The transmission of sound inside pipes. J Acoust Soc Am 11(3):205–210
6. Cremer L (1953) Theorie der Luftschall-Dämpfung im Rechteckkanal mit schluckender Wand und das sich dabei ergebende höchste Dämpfungsmaß. Acoustica 3:249–263
7. Nayfeh LJ, Kaiser E, Telionis Dp (1975) Acoustics of aircraft engine duct systems. AJAA J 13:130–153
8. Lapin AD (1975) Sound attenuation in waveguides (Review). Soviet P Acous 21:215–222

9. Büyükaksoy A, Demir A (2006) Radiation of sound from a semi-infinite rigid duct inserted axially into a larger infinite tube with in a cylindrical duct with wall impedance discontinuity. ZAMM 86:563–571
10. Rienstra SW (1992) Acoustical detection of obstructions in a pipe with a temperature gradient. In: Van der Burg A, Simonis J. (eds), Engineering mathematics modeling and methods. Kluwer, Dordrecht Holland, pp 151–179
11. Morse PM, Ingard K (1961) Linear acoustics theory. In: Flügge KHandbuch der Physiks, Akustik I. Springer, Berlin, Germany, pp 1–128
12. Rienstra SW, Hirschberg A (1992) An introduction to Acoustics. Instituut Wiskundige Dienst. [11, pp 151 – 179]. Kluwer, Dordrecht, Holland
13. Brekhovskikh L (1980) Waves in layered media, 2nd edn. Academic Press, New York
14. Rawlins AD (1995) A bifurcated circular waveguide problem. IMA J Appl Math 54:59–81
15. Noble B (1958) The Wiener–Hopf techique. Pergamon